Dynamic Partial Correlation Models

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EDS Brownbag Seminar
September 22 2022

Outline

Background - Approaches to modeling correlation matrices

- Consider a real-valued N-dimensional time series $\{\boldsymbol{y}_t\}_{t\in\mathbb{Z}}$ and a sequence of corresponding information sets $\mathcal{F}_{t-1} = \{\boldsymbol{y}_{t-1}, \boldsymbol{y}_{t-2}, \ldots\}$, satisfying

$$\boldsymbol{y}_t = \boldsymbol{\Sigma}_t^{1/2}\boldsymbol{\eta}_t, \qquad \boldsymbol{\eta}_t \sim \text{i.i.d.}(\mathbf{0}, \mathbf{I}_N), \tag{1}$$

and

$$\boldsymbol{\Sigma}_t = \boldsymbol{D}_t\boldsymbol{R}_t\boldsymbol{D}_t, \qquad \boldsymbol{D}_t = \text{diag}(\sigma_{i,i\,;\,t}, \ldots, \sigma_{N,N\,;\,t}), \qquad \boldsymbol{R}_t = (\rho_{i,j\,;\,t})_{i,j}$$

- We focus on modelling the dynamics of the conditional correlation matrix $\boldsymbol{R}_t$.
- One of the challenges is the parameterization of the dynamic $\boldsymbol{R}_t$.
- The matrix $\boldsymbol{R}_t$ has to be **positive definite with unit diagonal.**
- The first approach is that of Engle (2002):

$$\boldsymbol{R}_t = (\text{diag}\boldsymbol{Q}_t)^{1/2}\,\boldsymbol{Q}_t\,(\text{diag}\boldsymbol{Q}_t)^{1/2}, \qquad \boldsymbol{Q}_t = (1-\alpha-\beta)\boldsymbol{S} + \alpha\boldsymbol{u}_t\boldsymbol{u}_t^{\top} + \beta\boldsymbol{Q}_{t-1},$$

with $\boldsymbol{u}_t = \boldsymbol{D}_t^{-1}\boldsymbol{y}_t$, and $\alpha + \beta < 1$.

Some alternatives. . .

- A second approach casts the correlation matrix entries into hypersphere coordinates; see Rapisarda et al. (2007), Creal et al. (2011), and Buccheri et al. (2021).

Drawbacks. . .

- The correlation constraints are complicated.
- The stationarity and ergodicity conditions of these models are not well known.
- The asymptotic properties of MLE are unknown.

A possible solution. . .

- Recently, Archakov and Hansen (2021) introduced the possibility of modeling the strictly lower-half of the log-correlation matrix entries.
- The approach is extended to a dynamic setting by Hafner and Wang (2021) using score-driven dynamics.

However. . .

- All of these approaches treat the dynamics of $\boldsymbol{R}_t$ in its matrix form in order to ensure positive definiteness.
- A much more flexible approach would be to model each of the pairwise correlations separately.

⇒ **Problem**: This pairwise approach typically does not work as it need not produce a positive definite correlation matrix.

Our solution. . .

- **In this paper:** We solve this by looking at pairwise patterns of *partial correlations* using the work of Anderson (1958) and Joe (2006).
- **We need not worry about positive definiteness:** *Partial correlation coefficients* can be modeled independently and pairwise way in $(-1, 1)$.

Advantages:

- Computational and stability aspects, feasible theoretical and statistical properties, and significant gains in empirical performance.

The strategy: From partial correlations to correlation matrices

- A conditional partial correlation $\rho_{i,j|L_{ij}\,;\,t}$ for a set of indices L_{ij} with $i,j \notin L_{ij}$ is defined as the correlation between $\boldsymbol{y}_{i,t}$ and $\boldsymbol{y}_{j,t}$, conditional on $\mathcal{F}_{t-1}$ and on $\boldsymbol{y}_{L_{ij},t}$, where $\boldsymbol{y}_{L_{ij},t}$ is a vector containing the values of $\boldsymbol{y}_{k,t}$ for $k \in L_{ij}$.
- Define $\boldsymbol{V}_{i,j\,;\,t} = \rho_{i,j\,;\,t} - \boldsymbol{R}_{i,L_{ij};t}\boldsymbol{R}^{-1}_{L_{ij},L_{ij};t}\boldsymbol{R}_{L_{ij},j;t}$. The link between the pairwise and partial correlations is obtained from Anderson (1958) and Joe (2006):

$$\rho_{i,j|L_{ij}\,;\,t} = \frac{\rho_{i,j\,;\,t} - \boldsymbol{R}_{i,L_{ij};t}\boldsymbol{R}^{-1}_{L_{ij},L_{ij};t}\boldsymbol{R}_{L_{ij},j;t}}{\sqrt{\boldsymbol{V}_{i,i|L_{ij}\,;\,t}\cdot\boldsymbol{V}_{j,j|L_{ij}\,;\,t}}}, \tag{2}$$

for $i = 1,\ldots,N-1$, $j = i+1,\ldots,N$, and $L_{ij} = \{i+1,\ldots,j-1\}$, where

$$\mathbb{C}\text{orr}(\boldsymbol{y}_{i:j\,;\,t}) = \begin{bmatrix} 1 & \boldsymbol{R}_{i,L_{ij};t} & \rho_{i,j\,;\,t} \\ \boldsymbol{R}_{L_{ij},i;t} & \boldsymbol{R}_{L_{ij},L_{ij};t} & \boldsymbol{R}_{L_{ij},j;t} \\ \rho_{i,j\,;\,t} & \boldsymbol{R}_{j,L_{ij};t} & 1 \end{bmatrix}, \tag{3}$$

and $\boldsymbol{y}_{i:j\,;\,t} = (\boldsymbol{y}_{i,t},\ldots,\boldsymbol{y}_{j,t})^\top$.
- Inverting (2), we easily obtain the Pearson correlation:

$$\rho_{i,j\,;\,t} = \boldsymbol{R}_{i,L_{ij};t}\boldsymbol{R}^{-1}_{L_{ij},L_{ij};t}\boldsymbol{R}_{L_{ij},j;t} + \rho_{i,j|L_{ij}\,;\,t}\sqrt{\boldsymbol{V}_{i,i|L_{ij}\,;\,t}\cdot\boldsymbol{V}_{j,j|L_{ij}\,;\,t}}. \tag{4}$$

- The $N-1$ pairwise correlations and the $0.5(N-2)(N-1)$ partial correlations can vary independently in the interval $(-1, 1)$.
- By using the *D-Vine* structure proposed by Joe (2006), the resulting Pearson correlation matrix will **always** be **positive definite.**

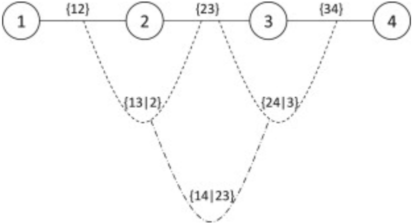


Figure: D-Vine

- **The key step:** We assume that $\boldsymbol{y}_t \mid \mathcal{F}_{t-1} \sim t\left(\mathbf{0}, \boldsymbol{R}_t, \nu\right)$, and so, for $j > i$ the distribution of $(\boldsymbol{y}_{i,t}, \boldsymbol{y}_{j,t})^\top \mid \boldsymbol{y}_{L_{ij},t}, \mathcal{F}_{t-1}$ with $\boldsymbol{y}_{L_{ij},t} = \{\boldsymbol{y}_{k,t}\}_{k \in L_{ij}}$, is Student's t

$$\begin{pmatrix} \boldsymbol{y}_{i,t} \\ \boldsymbol{y}_{j,t} \end{pmatrix} \mid \boldsymbol{y}_{L_{ij},t}, \mathcal{F}_{t-1} \sim t\left(\boldsymbol{\mu}_{i,j|L_{ij}\,;\,t} \,,\; \boldsymbol{D}^{1/2}_{i,j|L_{ij}\,;\,t} \boldsymbol{R}_{i,j|L_{ij}\,;\,t} \boldsymbol{D}^{1/2}_{i,j|L_{ij}\,;\,t} \,,\; \nu_{i,j|L_{ij}} \right), \quad (5)$$

where $\boldsymbol{R}_{i,j|L_{ij}\,;\,t}$ is the conditional partial (bivariate) correlation matrix

$$\boldsymbol{R}_{i,j|L_{ij}\,;\,t} = \begin{bmatrix} 1 & \rho_{i,j|L_{ij}\,;\,t} \\ \rho_{i,j|L_{ij}\,;\,t} & 1 \end{bmatrix},$$

$\nu_{i,j|L_{ij}} = \nu + \#L_{ij} = \nu + j - i - 1$ is the degrees of freedom,

$$\boldsymbol{\mu}_{i,j|L_{ij}\,;\,t} = \begin{pmatrix} \boldsymbol{R}_{i,L_{ij};t} \\ \boldsymbol{R}_{j,L_{ij};t} \end{pmatrix} \boldsymbol{R}^{-1}_{L_{ij},L_{ij};t}\, \boldsymbol{y}_{L_{ij},t}, \quad (6)$$

is the location, and

$$\boldsymbol{D}_{i,j|L_{ij}\,;\,t} = \frac{(\nu-2)(\nu + \boldsymbol{y}^\top_{L_{ij},t} \boldsymbol{R}^{-1}_{L_{ij},L_{ij};t} \boldsymbol{y}_{L_{ij},t})}{\nu \cdot \nu_{i,j|L_{ij}}} \begin{pmatrix} \boldsymbol{V}_{i,i|L_{ij}\,;\,t} & 0 \\ 0 & \boldsymbol{V}_{j,j|L_{ij}\,;\,t} \end{pmatrix} \quad (7)$$

a diagonal matrix; see Roth (2013) or Ding (2016).

Score recursions

- We use $\rho_{i,j|L_{ij}\,;\,t} = g(f_{i,j|L_{ij}\,;\,t})$ for $f_{i,j|L_{ij}\,;\,t} \in \mathbb{R}$, and define $\boldsymbol{y}_{i,j\,;\,t} = (\boldsymbol{y}_{i,t}, \boldsymbol{y}_{j,t})^\top$ for $j > i$ and let $p(\boldsymbol{y}_{i,j\,;\,t} \mid \boldsymbol{y}_{L_{ij},t}, \mathcal{F}_{t-1})$ be the Student's t pdf corresponding (5).
- Then we have the score expression

$$s_{i,j|L_{ij}\,;\,t} = \frac{\partial \log p(\boldsymbol{y}_{i,j\,;\,t} \mid \boldsymbol{y}_{L_{ij},t}, \mathcal{F}_{t-1})}{\partial f_{i,j|L_{ij}\,;\,t}} = \frac{1}{2} \boldsymbol{G}^\top_{i,j|L_{ij}\,;\,t} \left(\boldsymbol{R}^{-1}_{i,j|L_{ij}\,;\,t} \otimes \boldsymbol{R}^{-1}_{i,j|L_{ij}\,;\,t} \right) \times$$

$$\mathrm{vec}\left(w_{i,j|L_{ij}\,;\,t} \cdot \boldsymbol{D}^{-1/2}_{i,j|L_{ij}\,;\,t} \left(\boldsymbol{y}_{i,j\,;\,t} - \boldsymbol{\mu}_{i,j|L_{ij}\,;\,t} \right) \left(\boldsymbol{y}_{i,j\,;\,t} - \boldsymbol{\mu}_{i,j|L_{ij}\,;\,t} \right)^\top \boldsymbol{D}^{-1/2}_{i,j|L_{ij}\,;\,t} - \boldsymbol{R}_{i,j|L_{ij}\,;\,t} \right),$$

for $i = 1, \ldots, N-1$, $j = i+1, \ldots, N$, and $L_{ij} = \{i+1, \ldots, j-1\}$, with

$$w_{i,j|L_{ij}\,;\,t} = \frac{\nu_{i,j|L_{ij}} + 2}{\nu_{i,j|L_{ij}} + (\boldsymbol{y}_{i,j\,;\,t} - \boldsymbol{\mu}_{i,j|L_{ij}\,;\,t})^\top \boldsymbol{D}^{-1/2}_{i,j|L_{ij}\,;\,t} \boldsymbol{R}^{-1}_{i,j|L_{ij}\,;\,t} \boldsymbol{D}^{-1/2}_{i,j|L_{ij}\,;\,t} (\boldsymbol{y}_{i,j\,;\,t} - \boldsymbol{\mu}_{i,j|L_{ij}\,;\,t})},$$

$$\boldsymbol{G}_{i,j|L_{ij}\,;\,t} = \partial\, \mathrm{vec}(\boldsymbol{R}_{i,j|L_{ij}\,;\,t}) / \partial f_{i,j|L_{ij}\,;\,t} = \dot{g}\left(f_{i,j|L_{ij}\,;\,t} \right) \cdot \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}^\top .$$

- This leads to the score transition equation

$$f_{i,j|L_{ij}\,;\,t+1} = \omega_{i,j|L_{ij}} + \beta_{i,j|L_{ij}}\, f_{i,j|L_{ij}\,;\,t} + \alpha_{i,j|L_{ij}}\, s_{i,j|L_{ij}\,;\,t}. \tag{8}$$

Maximum likelihood estimation (MLE)

- Naturally, the true time-varying partial correlation processes $\{\rho_{i,j|L_{ij}\,;\,t}\}_{t\in\mathbb{Z}} = \{g(f_{i,j|L_{ij}\,;\,t})\}_{t\in\mathbb{Z}}$ are unobserved.
- As our model is observation driven, we can easily replace them by their initialized filtered counterparts $\{\hat{\rho}_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta})\}_{t=1}^{T} = \{g(\hat{f}_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta}))\}_{t=1}^{T}$.
- The likelihood is known in closed form as

$$\hat{L}_T(\boldsymbol{\theta}) = \sum_{t=1}^{T} \hat{\ell}_t(\boldsymbol{\theta}), \tag{9}$$

$$\hat{\ell}_t(\boldsymbol{\theta}) = \Bigg\{ \log \Gamma\left(\frac{\nu+N}{2}\right) - \log\Gamma\left(\frac{\nu}{2}\right) - \frac{N}{2}\log\left((\nu-2)\pi\right) - \frac{1}{2}\log|\hat{\boldsymbol{R}}_t(\boldsymbol{\theta})| + \frac{\nu+N}{2}\log\left(1+\frac{\boldsymbol{y}_t^\top \hat{\boldsymbol{R}}_t(\boldsymbol{\theta})^{-1}\boldsymbol{y}_t}{\nu-2}\right)\Bigg\},$$

where $\boldsymbol{\theta}$ contains ν, $\omega_{i,j|L_{ij}}$, $\alpha_{i,j|L_{ij}}$, $\beta_{i,j|L_{ij}}$, for $i=1,\ldots,N-1$ and $j=i+1,\ldots,N$, and $\{\hat{\boldsymbol{R}}_t(\boldsymbol{\theta})\}_{t=1}^{T}$ contains the filtered correlation matrices.
- The likelihood in (9) can be optimized numerically to yield the MLE

$$\hat{\boldsymbol{\theta}}_T = \arg\max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \hat{L}_T(\boldsymbol{\theta}). \tag{10}$$

Stationarity and ergodicity

- To establish stationarity and ergodicity of $\boldsymbol{y}_t$, we consider the model as a **DGP.**
- **The key assumption:** For $i = 1, \dots, N-1$ and $j = i+1, \dots, N$, let

$$\mathbb{E}\left[\log \max\left(\left|\beta_{i,j|L_{ij}} - \alpha_{i,j|L_{ij}} \cdot b_t\right| \;,\; \left|\beta_{i,j|L_{ij}} - \alpha_{i,j|L_{ij}} \cdot (1-\epsilon^2) \cdot b_t\right|\right)\right] < 0,$$

$b_t = \epsilon^2 \cdot (\frac{1}{2}(\nu_{i,j|L_{ij}} + 2)\tilde{b}_t - 1)$, and $\tilde{b}_t$ an i.i.d. sequence of Beta$(2, \nu_{i,j|L_{ij}})$ RVs.

Proposition (Strict Stationarity and Ergodicity)

Let $\hat{\boldsymbol{R}}_1$ denote a fixed initial correlation matrix with implied partial correlations $\hat{\rho}_{i,j|L_{ij}\,;\,1}$ and their transforms $\hat{f}_{i,j|L_{ij}\,;\,1}$. Then, the solutions $\hat{f}_{i,j|L_{ij}\,;\,t}$ of model (5)–(8) for $t \in \mathbb{N}$, initialized at $f_{i,j|L_{ij}\,;\,1}$ for $i = 1, \dots, N-1$, $j = i+1, \dots, N$, converge e.a.s. to unique strictly stationary and ergodic solutions $\{f_{i,j|L_{ij}\,;\,t}\}_{t\in\mathbb{Z}}$. In addition, the partial correlations $\hat{\rho}_{i,j|L_{ij}\,;\,t} = g(\hat{f}_{i,j|L_{ij}\,;\,t})$ and the Pearson correlations $\hat{\rho}_{i,j\,;\,t}$ converge e.a.s. to their unique stationary and ergodic limits $\{\rho_{i,j|L_{ij}\,;\,t}\}_{t\in\mathbb{Z}} = \{g(f_{i,j|L_{ij}\,;\,t})\}_{t\in\mathbb{Z}}$ and $\{\rho_{i,j\,;\,t}\}_{t\in\mathbb{Z}}$.

Invertibility

- To formulate the result, we need to introduce the demeaned and standardized bivariate observation vectors $\boldsymbol{y}^{\star}_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta})$ as

$$\boldsymbol{y}^{\star}_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta}) = \boldsymbol{D}_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta})^{-1/2}\left(\boldsymbol{y}_{i,j\,;\,t} - \boldsymbol{\mu}_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta})\right). \tag{11}$$

- These standardized observations make up the main input of the bivariate conditional Student's t distributions in (5).
- **They depend on the bivariate correlations** between the elements of $i, j \in L_{ij}$, i.e., on $\boldsymbol{R}_{L_{ij},L_{ij};t}$, which have to be estimated before the dynamics of $\rho_{i,j|L_{ij}\,;\,t}$.
- We therefore also introduce the **perturbed** counterparts $\hat{\boldsymbol{y}}^{\star}_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta})$ of $\boldsymbol{y}^{\star}_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta})$, where we replace the elements of $\boldsymbol{R}_{L_{ij},L_{ij};t}$ by those of $\hat{\boldsymbol{R}}_{L_{ij},L_{ij};t}$.
- We also distinguish three different filtered sequences:
  1. The filter $\hat{f}_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta})$, initialized at $\hat{f}_{i,j|L_{ij}\,;\,1}$ an taking $\hat{\boldsymbol{y}}^{\star}_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta})$.
  2. The filter $\hat{\hat{f}}_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta})$, initialized at $\hat{f}_{i,j|L_{ij}\,;\,1}$ but taking the SE $\boldsymbol{y}^{\star}_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta})$.
  3. The sequence $\{f_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta})\}_{t\in\mathbb{Z}}$, is the uninitialized SE limiting filter.

Invertibility

- The MLE procedure can only use the **perturbed** $\hat{\boldsymbol{y}}^{\star}(\boldsymbol{\theta})$ that use all previous pairs of correlation estimates, and produces $\hat{f}_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta})$ rather than $\hat{\hat{f}}_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta})$.
- Only for $j - i = 1$ we observe $\boldsymbol{y}^{\star}(\boldsymbol{\theta})$ directly.
- For $j - i = k > 1$, the score recursions for the filter also use the initialized sequence $\hat{f}_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta})$ for $j - i = 1, \ldots, k - 1$, which are not stationary and ergodic. As a result, we cannot apply Bougerol (1993).

⇒ **The way out:** If the filters $\hat{f}_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta})$ for $j - i < k$ converge exponentially fast and almost surely to their stationary and ergodic limits $f_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta})$, then we can use the results on perturbed SREs from Straumann and Mikosch (2006).

- **In summary:** To study the asymptotic properties of the MLE $\hat{\boldsymbol{\theta}}_T$, we need to study the stochastic limit properties of the filtered processes $\{\hat{f}_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta})\}_{t=1}^{T}$.

Invertibility

- **The key assumption:** The set $\boldsymbol{\Theta} \subset \mathbb{R}^d$ is a compact parameter space satisfying $\nu \geq 2 + \delta$ for some $\delta > 0$ and $\alpha_{i,j|L_{ij}} \neq 0$ for $i = 1, \ldots, N-1$ and $j = i+1, \ldots, N$, with

$$\mathbb{E}\left[\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \sup_{f} \log \left| \beta_{i,j|L_{ij}} + \alpha_{i,j|L_{ij}} \cdot \frac{\partial s_{i,j|L_{ij}\,;\,t}\left(f, \mathbf{y}^{\star}_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta}); \boldsymbol{\theta}\right)}{\partial f} \right|\right] < 0. \qquad (12)$$

Proposition (**filter invertibility**)

The filter processes $\{\hat{f}_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta})\}_{t\in\mathbb{N}}$ *initialized at fixed values* $\hat{f}_{i,j|L_{ij}\,;\,1}$ *converge exponentially fast almost surely to the unique stationary and ergodic sequences* $\{f_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta})\}_{t\in\mathbb{Z}}$ *uniformly over the parameter space* $\boldsymbol{\Theta}$*, that is*

$$\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \left|\hat{f}_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta}) - f_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta})\right| \xrightarrow{e.a.s.} 0,$$

$$\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \left|\hat{\rho}_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta}) - \rho_{i,j|L_{ij}\,;\,t}(\boldsymbol{\theta})\right| \xrightarrow{e.a.s.} 0,$$

$$\sup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \left|\hat{\rho}_{i,j\,;\,t}(\boldsymbol{\theta}) - \rho_{i,j\,;\,t}(\boldsymbol{\theta})\right| \xrightarrow{e.a.s.} 0,$$

as $t \to \infty$.

Asymptotic properties of the MLE

Theorem (Strong consistency and asymptotic normality of the MLE)

Under mild regularity conditions, the MLE is strongly consistent

$$\hat{\boldsymbol{\theta}}_T \xrightarrow{a.s.} \boldsymbol{\theta}_0 \qquad as \qquad T \to \infty,$$

and asymptotically normal

$$\sqrt{T}(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0) \Rightarrow \mathbb{N}(\mathbf{0}, \mathcal{I}^{-1}(\boldsymbol{\theta}_0)),$$

where $\mathcal{I}(\boldsymbol{\theta}_0)$ is the Fisher's Information matrix evaluated at the true parameter vector $\boldsymbol{\theta}_0$, that is $\mathcal{I}(\boldsymbol{\theta}_0) = -\mathbb{E}[\nabla^2 L(\boldsymbol{\theta})|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}]$.

Empirical illustration

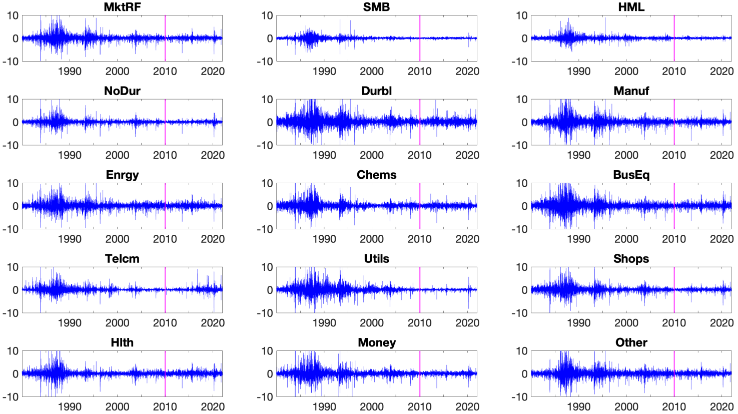


Figure: Daily returns on the three main risk factors and the twelve industry portfolios

Note: The period is 03 January 1980 to 31 December 2021. The vertical lines indicate the 4th of January 2010, i.e. the first trading day of 2010 and the start of the out-of-sample period.

Empirical illustration - In sample

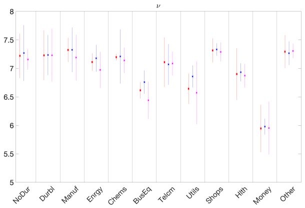
Note: the PCorr, t-GAS, and t-cDCC models are estimated over the sample 03 January 1980 to 31 December 2009. The log-Lik indicates the differences in log-likelihood value at the optimum. MSE and MAE relate to the differences in mean squared and mean absolute pricing errors, $e_t = r_{i,t} - \hat{\gamma}_{Mkt,t}(r_t^{Mkt} - r_t^F) - \hat{\gamma}_{SMB,t}SMB_t - \hat{\gamma}_{HML,t}HML_t$, where all return series are volatilty filtered.

	PCorr versus t-GAS			PCorr versus t-cDCC		
	log-Lik	MSE	MAE	log-Lik	MSE	MAE
NoDur	21.775	-0.001	-0.001	91.655	-0.001	-0.001
Durl	31.873	-0.002	-0.002	84.597	-0.001	-0.001
Manuf	21.858	-0.001	-0.002	64.218	-0.001	-0.001
Enrgy	-0.958	-0.004	-0.002	88.431	-0.004	-0.002
Chems	40.584	-0.003	-0.002	94.130	-0.002	-0.002
BusEq	-46.711	0.003	0.002	64.127	0.002	-0.001
Telcm	32.543	-0.003	-0.003	110.860	-0.001	-0.002
Utils	0.112	0.006	0.003	86.958	0.006	-0.003
Shops	28.824	-0.002	-0.002	83.624	-0.001	-0.001
Helt	23.673	-0.003	-0.001	91.652	-0.002	-0.002
Money	-30.370	0.003	0.002	76.455	-0.001	-0.002
Other	50.310	-0.002	-0.002	89.556	-0.001	-0.002

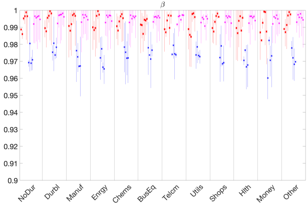
Empirical illustration - Out of Sample

This table contains the estimates of a_1^{Mod} for $Mod \in \{PCorr, t-GAS, t-cDCC\}$ in the regression model $r_{i,t} = a_0^{Mod} + a_1^{Mod} \cdot \hat{r}_{i,t}^{Mod} + u_{i,t}$, where $\hat{r}_{i,t}^{Mod}$ is obtained (recursively) using one-year-ahead estimates of $\boldsymbol{R}_t$ and $\gamma_{MKT,t}$, $\gamma_{SMB,t}$, and $\gamma_{HML,t}$. A $*$, $**$, or $***$ indicates rejection of $H_0 : a_0^{Mod} = 0$, $a_1^{Mod} = 1$, at the 10%, 5%, and 1% significance level, respectively. The MCS column indicates whether the model lies in the model confidence set of Hansen et al. (2011) based on tracking error MSE.

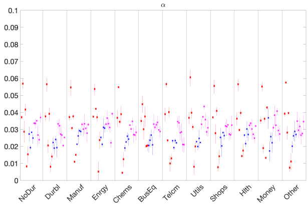
	PCorr		*t*-GAS		*t*-cDCC	
	$\hat{a}_1^{PCorr}$	*MCS*	$\hat{a}_1^{t-GAS}$	*MCS*	$\hat{a}_1^{t-cDCC}$	*MCS*
NoDur	1.013	✓	0.987		0.966 ***	
	(0.013)		(0.013)		(0.013)	
Durbl	1.018	✓	0.956 **		0.984	
	(0.013)		(0.013)		(0.012)	
Manuf	1.012	✓	1.001		0.975 ***	
	(0.007)		(0.007)		(0.007)	
Enrgy	1.053 **	✓	1.005		0.967 ***	
	(0.023)		(0.015)		(0.013)	
Chems	1.002	✓	0.981		0.965 ***	
	(0.011)		(0.012)		(0.011)	
BusEq	0.913 ***	✓	0.886 ***		0.861 ***	
	(0.006)		(0.007)		(0.006)	
Telcm	1.000	✓	0.975		0.945 ***	
	(0.014)		(0.014)		(0.013)	
Utils	1.050	✓	0.957		0.990	✓
	(0.032)		(0.023)		(0.020)	
Shops	0.987	✓	0.983		0.946 ***	
	(0.009)		(0.009)		(0.009)	
Hlth	1.009	✓	0.996 *		0.958 ***	
	(0.011)		(0.012)		(0.010)	
Money	0.986 *	✓	0.982 **	✓	0.928 ***	
	(0.006)		(0.006)		(0.007)	
Other	1.011	✓	1.010 *		0.974 ***	
	(0.006)		(0.006)		(0.006)	



(a) MLE of ν



(b) MLE of $\beta^{PCorr}_{i,j|L_{ij}}$, β^{DCC}_{i}, and $\beta^{GAS}_{i,j}$



(c) MLE of $\alpha^{PCorr}_{i,j|L_{ij}}$, α^{DCC}_{i}, and $\alpha^{GAS}_{i,j}$

Figure: Parameter estimates of all correlation models across industries

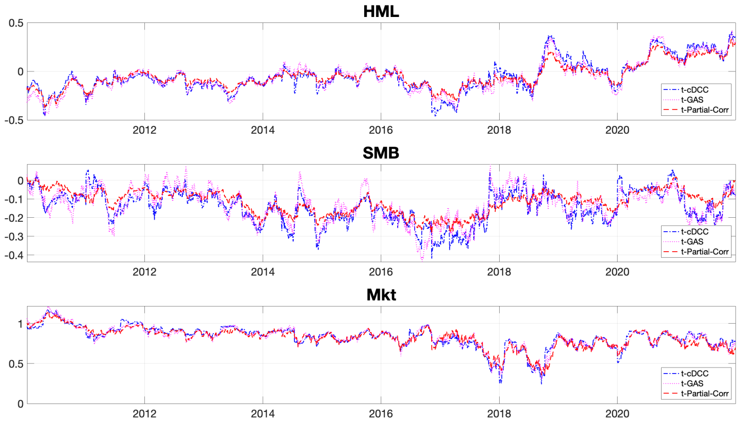


Figure: One-step ahead forecasts of the conditional betas of *NoDur*.

Concluding comments

⇒ Compare to other dynamic conditional correlation models, our dynamic partial correlation model introduced here has **several advantages:**

- Unlike the matrix equations in Creal et al. (2011), Opschoor et al. (2018, 2021), and Hafner and Wang (2021), there is **no complicated correlation constraint.**
- Precise **stationarity, ergodicity and invertibility conditions exist.**
- The parameters can be estimated recursively for a given value of ν, therefore, our set-up is **perfectly scalable to higher dimensions.**
- The **aymptotic theory of the MLE** is available.
- **The model works nicely in practice,** in particular for beta hedging.
- In a controlled simulation setting we show that **the new partial correlation model outperforms the considered benchmarks** Simulations

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Table: *MSE*, *MAE* and Frobenious norm simulation results for three different correlation models

Note: the labels PCorr, t-GAS and t-DCC indicate the new score-driven partial correlation model, the Student's t GAS model of Creal et al. (2011) with hypersphere parameterization, and the t-cDCC model of Engle (2002) with a multivariate Student's t log-likelihood, respectively. Results are based on 300 Monte Carlo experiments with sample size $T = 1000$ and $N = 4$. True correlation paths used in the data generating process are given from 100-day rolling window estimates of empirical correlation matrices of the series (HML, SMB, Mkt - RF, BusEq).

	MSE	*MAE*	*Frobenius*	*MSE*	*MAE*	*Frobenius*
	Gaussian			**Student** t_7		
PCorr	**0.0174**	**0.1036**	**0.4285**	**0.0192**	**0.1106**	**0.4543**
t-GAS	0.0264	0.1094	0.4324	0.0222	0.1204	0.4898
t-cDCC	0.0268	0.1177	0.4474	0.0273	0.1303	0.5386

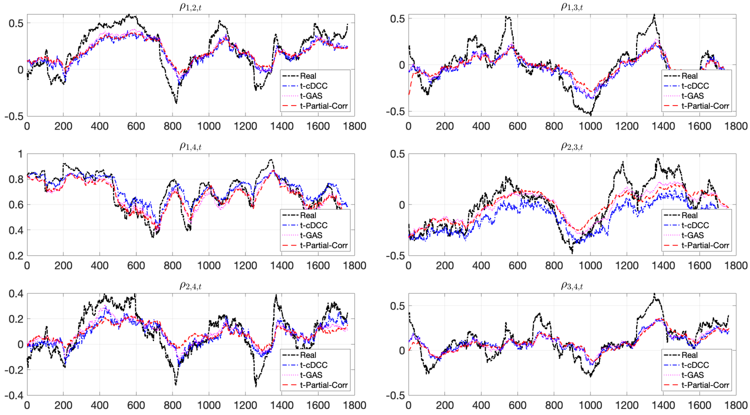

Figure: Comparison of the mean of the Monte Carlo simulation of the filtered conditional correlation coefficients with Student's t DGP with $\nu = 7$.

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